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Single-variable realisation of the SU(1, 1) spectrum generating algebra and discrete eigenvalue spectra of a class of potentials

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Abstract. A wide range of operators can be expressed in terms of the generators of the SU(1, 1) algebra, the spectral properties of which are well known. We attempt to find a realisation of the algebra in its most general form and thus evolve a unified approach to the problem of finding the spectra of Hamiltonians amenable to the technique rather than taking each case separately. We extend the analysis to obtain the spectra of potentials whose coordinate dependence is implicit.

1. Introduction

Numerous instances of fruitful applications of group theoretical methods, particularly in atomic and nuclear physics and the physics of fundamental interactions, have encouraged us to pursue this work.

We shall be concerned with the method of obtaining the eigenvalues of a certain class of dynamical systems by way of the SU(1, 1) Lie algebra. The non-relativistic single-particle Hamiltonian system is one such, and we have chosen to study this. Realisation of the generators of SU(1, 1) algebra in the function space of a single variable in terms of linear operators containing up to second-order derivatives have been used to find the spectrum of single-particle Hamiltonians with a class of one-dimensional potentials and three-dimensional central potentials [1–5]. Our attempt is to find such a realisation of the generators of SU(1, 1) in the most general form and to evolve a unified approach to the problem of finding the eigenvalue spectra of Hamiltonians amenable to the technique. In doing so, we are able to show that such a realisation is unique up to a similarity transformation and a transformation of the variable. The uniqueness supplements Lie's theorem [6], namely that the only finite-dimensional Lie algebras which can be realised in terms of vector fields in one variable are SL(2, R) and its subalgebras and the corresponding realisation is unique up to a change of variable.

More than reproducing the results that had been obtained earlier, this approach leads us to a class of what we call 'implicit potentials', by which we mean a potential function $V = V(\phi(x))$, where $\phi = \phi(x)$ is a function of the position variable x defined through an equation but cannot be expressed explicitly in terms of x.

2. Second-order realisation of SU(1, 1) algebra

We attempt to realise the SU(1, 1) Lie algebra

$$[\Gamma_1, \Gamma_2] = -\mathrm{i}\Gamma_3 \qquad [\Gamma_2, \Gamma_3] = \mathrm{i}\Gamma_1 \qquad [\Gamma_3, \Gamma_1] = \mathrm{i}\Gamma_2 \tag{2.1}$$

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in terms of second-order derivative operators in a single variable x. We write

$$\begin{aligned}\Gamma_1 &= a_1(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + c_1(x)\\ \Gamma_2 &= a_2(x)\frac{\mathrm{d}}{\mathrm{d}x^2} + b_2(x)\frac{\mathrm{d}}{\mathrm{d}x} + c_2(x)\\ \Gamma_3 &= a_3(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b_3(x)\frac{\mathrm{d}}{\mathrm{d}x} + c_3(x).\end{aligned} \tag{2.2}$$

A transformation of variable can be used to reduce one of the $a_i(x)$ to a predetermined non-zero constant. We set

$$a_1(x) = 1. \tag{2.3}$$

A similarity transformation

$$\exp\left(-\tfrac{1}{2}\int p(x)\,\mathrm{d}x\right)\Gamma_i \exp\left(\tfrac{1}{2}\int p(x)\,\mathrm{d}x\right) \tag{2.4}$$

can then be used to eliminate the first-order derivative term in one of the Γ_i. We set

$$b_1 = 0. \tag{2.5}$$

The conditions for realising the algebra (2.1) are then

$$\begin{aligned}& a_2' = a_3' = 0 \qquad b_2' = -\frac{\mathrm{i}}{2}a_3 \qquad b_3' = -\frac{\mathrm{i}}{2}a_2\\ & c_2' - a_2c_1' = -\frac{\mathrm{i}}{2}b_3 \qquad c_3' - a_3c_1' = -\frac{\mathrm{i}}{2}b_2 \qquad c_2'' - a_2c_1'' - b_2c_1' = -\mathrm{i}c_3\\ & c_3'' - a_3c_1'' - b_3c_1' = -\mathrm{i}c_2 \qquad a_3b_2' - a_2b_1' = -\frac{\mathrm{i}}{2}\\ & 2(a_3c_2' - a_2c_3') - (b_2b_3' - b_3b_2') = 0 \qquad (a_3c_2'' - a_2c_3'') + (b_3c_2' - b_2c_3') = -\mathrm{i}c_1.\end{aligned} \tag{2.6}$$

The general solution to this set of equations is

$$\begin{aligned}& a_2 = \alpha \qquad a_3 = \beta\\ & b_2 = -\frac{\mathrm{i}}{2}\beta(x+\gamma) \qquad b_3 = -\frac{\mathrm{i}}{2}\alpha(x+\gamma)\\ & c_1 = \frac{(x+\gamma)^2}{16} + \frac{\lambda}{(x+\gamma)^2} \qquad c_2 = -\frac{\mathrm{i}\beta}{4} - \alpha\left(\frac{(x+\gamma)^2}{16} - \frac{\lambda}{(x+\gamma)^2}\right)\\ & c_3 = -\frac{\mathrm{i}\alpha}{4} - \beta\left(\frac{(x+\gamma)^2}{16} - \frac{\lambda}{(x+\gamma)^2}\right)\end{aligned} \tag{2.7}$$

where γ and λ are arbitrary constants; and α and β are constants subject to the condition

$$\beta^2 - \alpha^2 = 1. \tag{2.8}$$

Without loss of generality, we set $\gamma = 0$ to get

$$
\begin{aligned}
\Gamma_1 &= \frac{d^2}{dx^2} + \frac{\lambda}{x^2} + \frac{x^2}{16} \\
\Gamma_2 &= \sinh\theta\left(\frac{d^2}{dx^2} + \frac{\lambda}{x^2} - \frac{x^2}{16}\right) + \cosh\theta\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}\right) \\
\Gamma_3 &= \cosh\theta\left(\frac{d^2}{dx^2} + \frac{\lambda}{x^2} - \frac{x^2}{16}\right) + \sinh\theta\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}\right)
\end{aligned}
\tag{2.9}
$$

where θ is an arbitrary constant. The Casimir invariant is

$$
\Gamma^2 \equiv \Gamma_3^2 - \Gamma_1^2 - \Gamma_2^2 = -\frac{3}{16} - \frac{\lambda}{4}. \tag{2.10}
$$

Now, to get the most general form of the realisation we invoke the similarity transformation (2.4) and the transformation of the variable implied in (2.3). The former leads to

$$
\begin{aligned}
\Gamma_1 &= \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) + \frac{x^2}{16} \\
\Gamma_2 &= \sinh\theta\left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) - \frac{x^2}{16}\right) + \cosh\theta\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}(1 + xp(x))\right) \\
\Gamma_3 &= \cosh\theta\left(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) - \frac{x^2}{16}\right) + \sinh\theta\left(-\frac{i}{2}x\frac{d}{dx} - \frac{i}{4}(1 + xp(x)\right)
\end{aligned}
\tag{2.11}
$$

where $p(x)$ is an arbitrary differentiable function and

$$
q(x) = \frac{p(x)}{4} + \tfrac{1}{2}\frac{dp(x)}{dx} + \frac{\lambda}{x^2}. \tag{2.12}
$$

To restore the variable transformation we simply put

$$
x = \phi(y)
$$

where y is the new variable and $\phi(y)$ is a differentiable function of y. This gives

$$
\begin{aligned}
\Gamma_1 &= u^2(y)\frac{d^2}{dy^2} + v(y)\frac{d}{dy} + w(y) + \frac{\phi^2(y)}{16} \\
\Gamma_2 &= \sinh\theta\left(u^2(y)\frac{d^2}{dy^2} + v(y)\frac{d}{dy} + w(y) - \frac{\phi^2(y)}{16}\right) \\
&\quad + \cosh\theta\left[-\frac{i}{2}\phi(y)\left(u(y)\frac{d}{dy} + \frac{p(y)}{2}\right) - \frac{i}{4}\right] \\
\Gamma_3 &= \cosh\theta\left(u^2(y)\frac{d^2}{dy^2} + v(y)\frac{d}{dy} + w(y) - \frac{\phi^2(y)}{16}\right) \\
&\quad + \sinh\theta\left[-\frac{i}{2}\phi(y)\left(u(y)\frac{d}{dy} + \frac{p(y)}{2}\right) - \frac{i}{4}\right]
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
u(y) &= \frac{1}{\phi'(y)} \qquad p(y) = \frac{v(y)}{u(y)} - u'(y) \\
w(y) &= \frac{p^2(y)}{4} + \frac{u(y)}{2}p'(y) + \frac{\lambda}{\phi^2(y)}.
\end{aligned}
\tag{2.14}
$$

Equation (2.13) is the most general form of the realisation of the SU(1, 1) algebra (2.1) in terms of second-order derivative operators in a single variable y. This realisation is described in terms of two arbitrary differentiable functions $\phi(y)$ (or $u(y)$) and $v(y)$ (or $p(y)$) which is a manifestation of the invariance of the algebra under variable transformations and similarity transformations, and two constants θ and λ. Of the two constants, λ has the significance of giving the Casimir invariant of the realisation (2.10) but θ has no physically significant meaning. This inessential constant may be got rid of by making a rotation about the 1 axis in group space

$$\Gamma_1 \to e^{-i\theta\Gamma_1}\Gamma_1\, e^{i\theta\Gamma_1} = \Gamma_1$$
$$\Gamma_2 \to e^{-i\theta\Gamma_1}\Gamma_2\, e^{i\theta\Gamma_1} = \cosh\theta\Gamma_2 - \sinh\theta\Gamma_2$$
$$\Gamma_3 \to e^{-i\theta\Gamma_1}\Gamma_3\, e^{i\theta\Gamma_1} = \cosh\theta\Gamma_3 - \sinh\theta\Gamma_3.$$

We may now recast the realisation (2.13) in its equivalent form

$$\begin{aligned}
\Gamma_1 &= u^2(y)\frac{d^2}{dy^2} + v(y)\frac{d}{dy} + w(y) + \frac{\phi^2(y)}{16} \\
\Gamma_2 &= -\frac{i}{2}\phi(y)\left(u(y)\frac{d}{dy} + \tfrac{1}{2}p(y)\right) - \frac{i}{4} \\
\Gamma_3 &= u^2(y)\frac{d^2}{dy^2} + v(y)\frac{d}{dy} + w(y) - \frac{\phi^2(y)}{16}.
\end{aligned} \tag{2.15}$$

This realisation is canonical in the sense that these operators are form invariant under variable transformations and similarity transformations. Any other realisation in second-order derivative operators in a single variable can be obtained as a special case of these. For example, taking $\phi(y) = y$ and $v(y) = 0$ we get the realisation of [1].

3. Unitary irreducible representation of SU(1, 1) Lie algebra and eigenvalues of differential operators

Unitary irreducible representations (UIR) of the SU(1, 1) Lie algebra in which the compact generator Γ_3 is diagonal can be classified according to the eigenvalues of Γ^2 and Γ_3 [4, 7, 8]. Denoting by q and m the eigenvalues of Γ^2 and Γ_3, respectively, we can write

$$q = j(j+1) \tag{3.1}$$

$$m = E_0 + N \tag{3.2}$$

where E_0 is a real number and $N = 0, \pm1, \pm2, \ldots$. The UIR of the SU(1, 1) algebra are classified into four series.

(a) Continuous principal series, $\mathscr{D}_p(q, E_0)$. Here $j = -\frac{1}{2} + i\rho$ where ρ is real and $q > \frac{1}{4}$ and

$$m = E_0, E_0 \pm 1, E_0 \pm 2, \ldots. \tag{3.3}$$

(b) Continuous supplementary series, $\mathscr{D}_s(q, E_0)$. Here $|j + \frac{1}{2}| < \frac{1}{2} - |E_0|$, j and E_0 are real, and

$$m = E_0, E_0 \pm 1, E_0 \pm 2, \ldots. \tag{3.4}$$

(c) Discrete series, $\mathscr{D}^+(j)$. Here j is real and $j<0$, and $E_0=-j$, and

$$m=-j, -j+1, -j+2, \ldots. \tag{3.5}$$

(d) Discrete series $\mathscr{D}^-(j)$. Here j is real and $j<0$, and $E_0=j$, and

$$m=j, j-1, j-2, \ldots. \tag{3.6}$$

In the discrete series $\mathscr{D}^+(j)$ the eigenvalue spectrum of Γ_3 is bounded from below while in $\mathscr{D}^-(j)$ it is bounded from above.

Now, the differential operator

$$\Gamma = u^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}+v(y)\frac{\mathrm{d}}{\mathrm{d}y}+w(y)-b\phi^2(y) \tag{3.7}$$

where $b\neq 0$ is a real constant and u, v, w, ϕ are related through (2.14), can be put into the form

$$\Gamma=(\tfrac{1}{2}-8b)\Gamma_1-(\tfrac{1}{2}+8b)\Gamma_3.$$

And the eigenvalue equation

$$\Gamma\Phi(y)=\mu\Phi(y) \tag{3.8}$$

can be transformed by a suitable rotation about the 2 axis to the eigenvalue equation [4]

$$\Gamma_1\Phi_1(y)=\frac{\mu}{4\sqrt{-b}}\Phi_1(y) \tag{3.9}$$

or to the equation

$$\Gamma_3\Phi_3(y)=\frac{\mu}{4\sqrt{b}}\Phi_3(y). \tag{3.10}$$

The former gives continuous eigenvalues as Γ_1 is the non-compact generator while in the latter case we get a discrete spectrum. This (latter) case yields the eigenvalue

$$\mu=4\sqrt{b}(E_0+N) \qquad N \text{ is an integer} \tag{3.11}$$

the real number E_0 depending on the class of UIR of the algebra, and hence on the Casimir eigenvalue

$$j(j+1)=-\frac{3}{16}-\frac{\lambda}{4}$$

or

$$j=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{\tfrac{1}{4}-\lambda} \qquad \text{real for } \lambda<\tfrac{1}{4}. \tag{3.12}$$

If j is real and negative, the spectrum of μ belongs to the discrete class $\mathscr{D}^{\pm}(j)$ with $E_0=\mp j$. Hence for

$$-\tfrac{3}{4}<\lambda<\tfrac{1}{4} \qquad \text{and} \qquad j=-\tfrac{1}{2}(\mathrm{i}\pm\sqrt{\tfrac{1}{4}-\lambda})$$

and for

$$\lambda\leqslant-\tfrac{3}{4} \qquad \text{and} \qquad j=-\tfrac{1}{2}(1+\sqrt{\tfrac{1}{4}-\lambda}) \tag{3.13}$$

we have

$$\mu=\pm4\sqrt{b}(N-j) \qquad N=0,1,2,\ldots. \tag{3.14}$$

The upper (lower) sign in (3.14) holds for a spectrum bounded from below (above).

4. Schrödinger eigenvalue problems solvable through the UIR of SU(1, 1) algebra

Eliminating the first-order derivative term in the eigenvalue equation (3.8), namely

$$\left[u^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}+v(y)\frac{\mathrm{d}}{\mathrm{d}y}+w(y)-b\phi^2(y)-\mu\right]\Phi(y)=0 \tag{3.8}$$

and dividing by u^2, we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{1}{u^2}\left(\frac{u'^2}{4}-\frac{uu''}{2}+\frac{\lambda}{\phi^2}-b\phi^2-\mu\right)\right]\psi(y)=0. \tag{4.1}$$

This equation reduces to the Schrödinger eigenvalue equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}+2(E-V(y))\right)\psi(y)=0 \tag{4.2}$$

if the non-derivative term in the operator in (4.1), to be denoted by $Q(y)$ as

$$\begin{aligned} Q(y)&=\frac{1}{u^2}\left(\frac{u'^2}{4}-\frac{uu''}{2}+\frac{\lambda}{\phi^2}-b\phi^2-\mu\right)\\ &=\frac{\phi'''}{2\phi'}-\frac{3}{4}\left(\frac{\phi''}{\phi'}\right)^2+\phi'^2\left(\frac{\lambda}{\phi^2}-b\phi^2-\mu\right)\end{aligned} \tag{4.3}$$

can be reduced to the form

$$Q(y)=2(E-V(y)) \tag{4.4}$$

by a suitable choice of the function $\phi(y)$ or $u(y)$. This will enable us to find the eigenvalue E in the Schrödinger equation (4.2). There are seven possible cases in which the last term in expression (4.3) can yield a constant to be identified as $2E$, thereby leading to the desired form of $Q(y)$. These cases correspond to specific properties of the function $p(y)$ defined through a first-order differential equation. Below, we elaborate, considering only the discrete spectrum.

(a) For the choice

$$\phi'(y)=\text{constant} \qquad \text{i.e. } \phi(y)=y \tag{4.5}$$

we find

$$Q(y)=-\mu-\left(by^2-\frac{\lambda}{y^2}\right).$$

That is, for a potential function $V(y)$ of the Kratzer [8] type

$$2V(y)=Ay^{-2}+By^2 \qquad B>0 \tag{4.6}$$

the Schrödinger equation (4.2) has the eigenvalue spectrum

$$E=-\tfrac{1}{2}\mu=\pm2\sqrt{B}\left(N+\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{\tfrac{1}{4}-A}\right). \tag{4.7}$$

(b) For the choice

$$\phi'(y)=(2\phi)^{-1} \qquad \text{i.e. } \phi(y)=\sqrt{y} \tag{4.8}$$

we find

$$Q(y)=-\frac{b}{4}-\left[\frac{\mu}{4}y^{-1}-\left(\frac{\lambda}{4}+\frac{3}{16}\right)y^{-2}\right].$$

That is, for the potential function

$$V(y)=\frac{K}{y}-\frac{L}{y^2} \tag{4.9}$$

the discrete spectrum of E is obtained as

$$\begin{aligned} E&=-\frac{b}{8}=-\frac{1}{8}\frac{\mu^2}{16(N-j)^2}\\ &=-\frac{K^2}{2(N+\frac{1}{2}\pm\frac{1}{2}\sqrt{8L+1})^2}. \end{aligned} \tag{4.10}$$

(c) Next, we choose $\phi(y)$ to satisfy the equation

$$\phi'(y)=-\alpha\phi(y) \tag{4.11}$$

where α is a positive constant. With this choice

$$\phi(y)=\mathrm{e}^{-\alpha y}$$

and

$$Q(y)=\alpha^2(\lambda-\tfrac{1}{4})-\alpha^2(b\,\mathrm{e}^{-4\alpha y}+\mu\,\mathrm{e}^{-2\alpha y}).$$

So, for the Morse potential

$$V(y)=A\,\mathrm{e}^{-4\alpha y}-B\,\mathrm{e}^{-2\alpha y}\qquad A,B>0 \tag{4.12}$$

we obtain, setting $b=2A/\alpha^2$ and $\mu=-2B/\alpha^2$, the eigenvalues

$$\begin{aligned} E&=-\frac{\alpha^2}{2}(\tfrac{1}{4}-\lambda)\\ &=2\alpha^2\left(N+\tfrac{1}{2}-\frac{B}{2\alpha\sqrt{2A}}\right)^2\qquad N=0,1,2,\ldots \end{aligned} \tag{4.13}$$

where λ has been determined from (3.14) with the lower sign, as μ is negative ($B>0$).

(d) For $\phi(y)$ satisfying the equation

$$\phi'(y)=\frac{1}{\sqrt{\phi^2+1}} \tag{4.14}$$

we get

$$Q(y)=\frac{1}{(\phi^2+1)^2}\left(\frac{-5/4}{\phi^2+1}+\frac{\lambda}{\phi^2}+\lambda+\tfrac{3}{4}\right)-b\frac{\phi^2+\mu/b}{\phi^2+1}.$$

Imposing the condition $\mu=b$, the above reduces to the form of (4.4) with

$$E=-\tfrac{1}{2}b \tag{4.15}$$

and

$$V(y)=V(\phi(y))=\frac{1}{2(\phi^2+1)^2}\left(\frac{5/4}{\phi^2+1}-\frac{\lambda}{\phi^2}-(\lambda+\tfrac{3}{4})\right). \tag{4.16}$$

Equations (3.14) and (4.15) together with the condition $\mu=b$ give

$$\begin{aligned} E&=-8(N-j)^2\\ &=8(N+\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{\tfrac{1}{4}-\lambda})^2\qquad N=0,1,2,\ldots \end{aligned} \tag{4.17}$$

the lower sign being applicable only for $-\frac{3}{4}<\lambda\leq\frac{1}{4}$. In other words, for a potential of the form of (4.16) where $\phi(y)$ is a solution of (4.14) the Hamiltonian $H=-\frac{1}{2}\mathrm{d}^2/\mathrm{d}y^2+V(y)$ has the eigenvalue spectrum given by (4.17).

Table 1. Eigenvalue spectrum of Schrödinger operator $-\frac{1}{2}\mathrm{d}^2/\mathrm{d}y^2 + V(y)$ solvable through the realisation of the SU(1, 1) algebra.

	Equation for ϕ	Form of potential $V(y)$	Eigenvalue spectrum E_N, $N = 0, 1, 2, \ldots$
(a)	$\phi'(y) = 1$	$Ay^{-2} + By^2$	$\pm\sqrt{2B}(N + \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{4} + 2A})$
(b)	$\phi'(y) = (2\phi)^{-1}$	$Ay^{-1} + By^{-2}$	$\dfrac{-A^2}{2(N + \frac{1}{2} \pm \sqrt{8B+1})}$
(c)	$\phi'(y) = -\alpha\phi$	$A\,\mathrm{e}^{-4\alpha y} - B\,\mathrm{e}^{-2\alpha y}$	$-2\alpha^2\left(N + \frac{1}{2} - \dfrac{B}{2\alpha\sqrt{2A}}\right)^2$
(d)	$\phi'(y) = (\phi^2+1)^{-1/2}$	$\dfrac{5}{8(\phi^2+1)^3} - \dfrac{\lambda}{2\phi^2(\phi^2+1)^2} - \dfrac{\lambda + \frac{3}{4}}{2(\phi^2+1)^2}$	$-8(N + \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda})^2$
(e)	$\phi'(y) = \dfrac{\phi(y)}{\sqrt{1-\alpha\phi^2}}$	$\dfrac{b\phi^4}{2(1-\alpha\phi^2)} - \dfrac{\alpha\phi - \frac{1}{4}}{2(1-\alpha\phi^2)^3}$	$\dfrac{1}{4a^2}\left(\mp 2(N+\frac{1}{2})a - \frac{1}{4} \pm \sqrt{\pm(N+\frac{1}{2})a + \frac{1}{4}a^2 + \frac{1}{16}}\right)$
(f)	$\phi'(y) = \dfrac{\phi}{\sqrt{1-\alpha\phi^4}}$	$\dfrac{\mu\phi^2}{1-\alpha\phi^4} - \dfrac{\frac{3}{4}\alpha^2\phi^8 + \frac{9}{2}\alpha\phi^4 - \frac{1}{4}}{2(1-\alpha\phi^4)^3}$	E_N are given by the solutions of the equation $\left(E^2 + 2(N^2 + N + \frac{3}{16})E + \dfrac{\mu^2}{16\alpha}\right)^2 = \dfrac{\mu^2}{2\alpha}(N+\frac{1}{2})^2 E$
(g)	$\phi'(y) = \dfrac{\phi}{\sqrt{\phi^4 + \alpha\phi^2 + \beta}}$	$-\dfrac{3\phi^8 - 2\alpha\phi^6 - 18\beta\phi^4 - 6\alpha\beta\phi^2 - \beta^2}{8(\phi^4 + \alpha\phi^2 + \beta)^3}$	$\dfrac{N^2 + N + \frac{3}{16}}{(\frac{1}{8}\alpha^2 - \frac{1}{2}\beta)} \pm \dfrac{\alpha(N+\frac{1}{2})\sqrt{\frac{1}{16}(\frac{1}{16}\alpha^2 - \frac{1}{4}\beta) + \frac{1}{4}\beta(N+\frac{1}{2})^2}}{(\frac{1}{8}\alpha^2 - \frac{1}{2}\beta)^2}$

The above results and few other choices of $\phi'(y)$ are collected in a tabular form in table 1. In cases (a), (b) and (c), the potentials can be expressed in terms of the position variable y. The potential function in (4.6) is the Kratzer potential [9] encountered in molecular physics. In nuclear theory this type of potential occurs in the two-rotor model of deformed nuclei [10]. The special case $A=0$ yields the one-dimensional harmonic oscillator potential with force constant $k=B$ for which the allowed spectrum is given by $E=2\sqrt{k}(N+\frac{1}{2}\pm\frac{1}{4})$ which can also be written in the usual form $E_n=\omega(n+\frac{1}{2})$, $n=0,1,2,\ldots$, ω being the classical frequency $\sqrt{k}$, ($\hbar=m=1$).

The three-dimensional isotropic harmonic oscillator also falls under the case (a), with $A=l(l+1)$ and $B=k$ giving energy eigenvalues $E=\sqrt{k}(2N+l+1\pm\frac{1}{2})$, the lower sign inside the parenthesis applicable only for $l=0$ (see (3.13) and note that $\lambda=-\frac{1}{2}l(l+1)$). With $n=2N+l$ the energy eigenvalues can be written as $E_n=\omega(n+\frac{3}{2})$ for $l\neq 0$ and $E_n=\omega(n+\frac{1}{2})$ for $l=0$. Here, the case $l=0$ coincides with that of the linear harmonic oscillator, implying the existence of an eigenvalue $E_0=\frac{1}{2}\omega$ even for the three-dimensional oscillator, contrary to the well known result [11]. In spite of the subtle group theoretical analysis of the quantum oscillator problem [12], we do not see any simple way of ruling out this eigenvalue. However, on the elementary level of non-relativistic quantum mechanics it is easily ruled out because the wavefunction corresponding to this eigenvalue is not normalisable because of a pole at the origin.

In case (b) the potential function (4.9) has two terms—the first term corresponds to an inverse square force and the second to an inverse cubic force. It thus covers the generalised Kepler problem including the non-relativistic hydrogen atom. With $K=-Ze^2$ and $L=\frac{1}{2}l(l+1)$ the potential $V(y)$ is the effective Coulomb potential of the hydrogen-like atom and the discrete energy spectrum can be written as $E_n=-Z^2e^4/(2n^2)$ where $n=N+l+1$, a positive integer. Degeneracy of each level is also correctly reproduced. The potential function (4.12) is the well known Morse potential [13] when $B=2A$.

In cases (d) and (e)-(g) (given in table 1) the equations for $\phi(y)$ cannot be solved to give ϕ in terms of y. For example, (4.14) has the solution

$$y=\tfrac{1}{2}\phi\sqrt{1+\phi^2}+\tfrac{1}{2}\sinh^{-1}\phi$$

which cannot be inverted to give ϕ in terms of y. Such potentials we call, somewhat arbitrarily, 'implicit potentials' though the term may not be quite appropriate. We fail to recognise physical significance, if any, of such potentials except for a vague hunch that the equation for $\phi(y)$ might remotely be some sort of constraint or condition for non-relativistic string-like systems having some internal structure. It is beyond the purview of the present study to delve into this remote possibility. We present these cases simply because they emerge in the unified approach we have adopted. At the most, these rather murky cases are exercises in which we have solutions looking for physical problems.

Before we conclude this paper we note that we have confined ourselves to discrete spectra. It may be noted that analysis of scattering states for the one-dimensional Morse potential and the Poschl–Teller potential [14] by constructing two-dimensional realisations of SU(1, 1) on the unit hyperboloid had been carried out in [15].

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